PHILOSOPHICAL TRANSACTIONS.

I. *Some Measurements of Atmospheric Turbulence.*

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[PLATE 1.]

CONTENTS.

I. NOTATION.

THE following notation is used throughout. The co-ordinate axes are a right-handed rectangular system x, y, h, in which $0h$ is directed vertically upwards, and $0x$ lies in any azimuth which happens to be convenient. Elements of distance to east and to north are denoted by de, dn, so that they are special cases of dx, dy. The atmospheric density is ρ, the pressure is p, acceleration of gravity is g, latitude ϕ is reckoned negative in the southern hemisphere, and ω is the angular velocity of the earth. Velocities are denoted by v with a suffix to indicate the direction *towards* which they blow. Momenta per unit volume are denoted by m_X, m_Y, m_H. The eddy-diffusivity is denoted by a capital K as in G. I. TAYLOR'S recent papers. Another, and in the author's opinion a better, measure of turbulence is ξ discussed in a previous paper.* The relation K to ξ is given by

$$\frac{\partial}{\partial p}\left(\xi\frac{\partial\chi}{\partial p}\right) = \frac{\partial\chi}{\partial t} = \mathrm{K}\frac{\partial^2\chi}{\partial h^2} \quad \ldots \quad (1)$$

where χ is either potential temperature, or else mass of water or smoke per mass of atmosphere. If ρ and ξ were independent of height, then from (1) we should have

$$\xi = g^2\rho^2\mathrm{K}. \quad \ldots \quad (2)$$

* L. F. RICHARDSON, 'Roy. Soc. Proc.,' A, vol. 96 (1919), pp. 9 to 13.

It is suggested that ξ might be named "the turbulivity." Its dimensions are: $(\text{mass})^2 \times (\text{length})^{-2} \times (\text{time})^{-5}$.

The advantage of using ξ instead of K is that the former enables one to allow for variations of density and of turbulence in a simple and natural manner. The disadvantage of ξ is that it has no name derivable from indoor physics. It is suited to the free atmosphere. We might compromise by using in place of K or ξ the "eddy-conductivity," c, defined by the equation

$$\frac{\partial\,(\rho\chi)}{\partial t} = \frac{\partial}{\partial h}\left(c\frac{\partial\chi}{\partial h}\right) \text{ or, approximately, } \frac{\partial\chi}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial h}\left(c\frac{\partial\chi}{\partial h}\right). \quad . \quad . \quad . \quad (3)$$

In so doing we gain an acceptable name "conductivity," but we lose by the explicit appearance of density in the equation. Either c or ξ allows variations of turbulence with height to be treated correctly, while K does not do so, as has been pointed out elsewhere by the author.* The dimensions of c are $(\text{mass}) \times (\text{length})^{-1} \times (\text{time})^{-1}$.

This c is of the same dimensions as the measure of turbulence discussed by W. SCHMIDT, of Vienna, under the name of "Austausch" in two important papers. ('Sitz. Akad. Wiss.,' Wien, 1917 and 1918.)

However much turbulence and density may vary with height

$$g^2\rho c = \xi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On the contrary if there are no variations with height,

$$c = \rho\text{K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The six components of stress are denoted by $\widehat{xx}$, $\widehat{yy}$, $\widehat{hh}$, $\widehat{xy}$, $\widehat{yh}$, $\widehat{hx}$, as in the writings of K. PEARSON.

The convention adopted for the signs of eddy-stresses conforms to that of LOVE's "Theory of Elasticity." Tractions are reckoned positive. That is to say, a direct stress such as $\widehat{xx}$ is positive if it be a tension, negative if a pressure; and a shearing stress such as $\widehat{xh}$ is positive when the air on that side of a level surface for which h is greater (*i.e.*, above), drags the air below in the sense of x increasing.

The *definition of eddy-viscosity* adopted in this paper is

$$\frac{\text{eddy shearing stress}}{\text{rate of mean shearing strain}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{A})$$

in agreement with the definition used by W. SCHMIDT (*loc. cit.*, 1917, p. 5).

The advantage of this definition is that it is simply based on the fundamental ideas of stress and strain, as well as being in harmony with the definition adopted in the theory of viscous liquids. (*cf.*, LAMB, 'Hydrodynamics,' IV. edn., § 326).

The question may arise as to whether the viscosity defined by (A) can ever become infinite by the vanishing of the denominator. The point is discussed by the author

* *Loc. cit.*

in the paper already cited (p. 13), and the conclusion is reached that such an occurrence would be highly improbable.

The relation of ξ to the eddy-viscosity is most easily reached *via* terms $\partial m_X/\partial t$, $\partial m_Y/\partial t$ in the dynamical equations. If pressure-gradient just balanced geostrophic wind we should have

$$\frac{\partial\,(\widehat{xh})}{\partial h}=\frac{\partial m_X}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now by the definition of viscosity given above

$$\widehat{xh}=\mu\,.\,\partial v_X/\partial h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting (7) in (6) and inserting $m_X=\rho v_X$ there results

$$\frac{\partial}{\rho\,\partial h}\left\{\mu\,\frac{\partial v_X}{\partial h}\right\}=\frac{\partial v_X}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It is seen that this equation becomes identical with (1) if

$$\chi=v_X \qquad \text{and} \qquad \xi=g^2\rho\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

of which the latter is the required relation.

On comparing equations (8) and (3), it is seen that eddy-viscosity, μ, and eddy-conductivity, c, are of the same dimensions, and appear in their respective differential equations in the same way. Indeed, TAYLOR has suggested that they are equal.* This likeness would be a good argument for recording observations in terms of these two quantities instead of in terms of diffusivity K or turbulivity ξ.

II. Shearing Stress from Pilot Balloon Observations.

(*Condensed and revised January 22, 1920.*) EKMAN† in a remarkable paper pointed out that the total momentum of water produced by a tangential stress on the surface of the sea, in the steady state, is directed at right angles to the tangential stress, and its amount is quite independent of the value of the viscosity or of the variation of viscosity with depth. The same applies to the atmosphere. We may use this principle to find the shearing stress on the ground, provided we have a measure of what the momentum would be if the surface stress were zero.

I have taken the wind at a height of $1\frac{1}{2}$ km. to $2\frac{1}{2}$ km. as the standard of reference, because, by so doing, the term depending on curvature of path, and other small terms in the dynamical equations, are automatically allowed for to a first approximation. The stress at 2 km. is undoubtedly much less than that on the ground, and is neglected. It is best to select observations in which the momentum becomes nearly independent of height above 1·5 km. A table of results follows. They were computed with the help of Mrs. L. F. RICHARDSON. Dr. H. JEFFERYS says the selection will select abnormal lapse-rates and so abnormal viscosities.

* 'Phil. Trans.,' A, vol. 215, p. 22.

† "On the Influence of the Earth's Rotation on Ocean Currents," by V. W. EKMAN, 'Arkiv for Matem. Astr. och Fysik,' Stockholm, Bd. II., No. 11 (1905).

TABLE I.—Shearing Stresses at the Earth's Surface.

Here $\overline{m}$ is the mean momentum per cm.^{3} of the air between the ground and a height 2 km. above sea level.

Place.	Latitude.	$\overline{m}$. $\frac{\text{grm.}}{\text{cm.}^2\ \text{sec.}}$.	Resultant stress $\div\ \overline{m}^2$. $\frac{\text{cm.}^3}{\text{grm.}}$.	Stress on ground is veered* from wind near surface.	Stress dragging ground in the direction of $\overline{m}$. dynes cm.$^{-2}$.	Stress dragging ground perpendicularly to left of $\overline{m}$.	Reference and notes.
Eskdalemuir .	55·3°	0·72	6·7	− 8°	+ 2·9	+ 1·9	Mean of 39 observations in 1913 and 1914. Light winds omitted.
Lindenberg .	52·2	1·24	1·0	26°	+ 1·5	+ 0·05	General mean. E. GOLD, Met. Office, 'Geophys. Mem.,' V., p. 143.
Upavon . .	51·3	1·70	1·0	assumed to be zero	+ 2·9	+ 0·8	Strong winds. DOBSON, 'Q. J. Met. Soc.,' April, 1914.
Batavia . . .	− 6·2	0·26	1·3	− 21°	+ 0·08	− 0·03	May and June . . . . VAN BEMMELEN; Batavia, 'Verhandelingen,' vol. 1, 1911, pp. 13, 14.
		0·47	1·1	30°	+ 0·04	− 0·24	July to September . .
		0·47	0·6	± 180°	− 0·12	− 0·05	December to February .
The above were obtained by the method described in the preceding page. Below follows a result taken from G. I. TAYLOR'S paper, 1915.							
Upavon . . .	51·3	as above	0·83	assumed to be zero	—	—	The same observations as above.

* Veering here means turning in the sense of north to east, and this statement applies to both sides of the equator.

III. Eddy-viscosity from Pilot Balloon Observations.

(*Abridged, January 22, 1920.*) The method of the last section will give the difference of the shearing stresses on two surfaces of any horizontal slab of air. If we choose one of the surfaces so that $\partial v/\partial h = 0$ and consequently also the stress vanishes, we obtain the stress on the other surface. This has been done for some very smooth means for Lindenberg (E. Gold, Met. Office, 'Geophys. Mem.,' V., p. 143). The results are set out in the following table :—

Table II.—Lindenberg.

The x axis is directed with the surface wind. The stress is that exerted by the upper on the lower layer.

Height above mean sea, kilometres.	Eddy-shearing-stresses.		Rates of mean shearing.		Eddy viscosities.	
	$\widehat{xh}$ dynes cm.$^{-2}$.	$\widehat{yh}$ dynes cm.$^{-2}$.	$\frac{\partial v_x}{\partial h}$ sec.$^{-1}$ $10^3\times$	$\frac{\partial v_y}{\partial h}$ sec.$^{-1}$ $10^3\times$	Parallel to wind. $\frac{\widehat{xh}}{\partial v_x/\partial h}$ dyne cm.$^{-2}$ sec.	Perpendicular to wind. $\frac{\widehat{yh}}{\partial v_y/\partial h}$ dyne cm.$^{-2}$ sec.
1·0	+0·05	−0·09	− 1·0	− 0·2	50	450
0·8	+0·01	−0·37	− 1·2	− 1·5	10	250
0·7	zero					
0·6	+0·02	−0·71	1	− 3·2	20	220
0·4	+0·09	−0·99	6·0	− 9·0	15	110
0·3	+0·39	−1·06	12·5	−12·2	31	87
0·2	+0·69	−0·90				
0·12 ground	+1·10	−0·64	21·5 ?	−17·5 ?	51	37

To obtain a quantity comparable with ξ we must multiply the eddy-viscosity by $g^2\rho$ which is approximately 1100 c.g.s. units.

The mean of the viscosities in the two directions increases with height as we might expect from other observations (*vide* Part VIII., below, also 'Roy. Soc. Proc.,' A, vol. 96 (1919), p. 18). But the most interesting thing about this table is the *marked lack of isotropy in viscosity. The air appears to be more viscous, for large motions, across the wind than parallel to it, except just near the ground.*

IV. Eddy-diffusivity from Smoke or Floating Bodies.

Some direct measurements have been made by observing the gradually increasing scatter of smoke or other visible material carried along by the air. The changes in height of a large number of small portions of air are observed during a fixed interval of time. These changes are found to be distributed about their mean value

approximately according to the ordinary "law of error." Their scatter in height is measured by the "standard deviation," computed by the familiar methods.* In order to find the diffusivity K the observations are compared with an appropriate integral of the approximate equation

$$\frac{\partial \chi}{\partial t} = \mathrm{K}\frac{\partial^2 \chi}{\partial h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Such a one is

$$\chi = \frac{\mathrm{A}_1}{\sqrt{4t+\mathrm{A}_2}}\, e^{\frac{-(h-\mathrm{A}_3)^2}{\mathrm{K}(4t+\mathrm{A}_2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where A_1, A_2, A_3 are constants.

This integral represents a horizontal lamina in which the density χ is distributed about a mean height A_3 according to the law of error. The square of the standard deviation of the mass in the lamina can be shown to be

$$(4t+\mathrm{A}_2)\frac{\mathrm{K}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

So if the scatter of the same set of particles be observed at the beginning and at the end of an interval T of time, it follows that

$$\mathrm{K} = \frac{1}{2\mathrm{T}}\text{ (increase during T of square of standard deviation).} \quad . \quad . \quad . \quad (4)$$

But the increase of the square of the standard deviation is equal to the square of the standard deviation of the change of height. Accordingly

$$\mathrm{K} = \frac{1}{2\mathrm{T}}\text{ (square of standard deviation of change of height during T).} \quad . \quad . \quad (5)$$

In this last transformation we have assumed that K is sensibly independent of height. This is permissible because the range of scatter can usually be made small. For the same reason the density of the air may be taken as independent of the height, so that we may obtain from K, the constant ξ, which we require when pressure is taken as independent variable in place of height, in accordance with (1) above. This procedure is not perfectly satisfactory but it is very convenient. It gives $\xi = g^2\rho^2\mathrm{K}$ and "eddy-conductivity" $= \rho\mathrm{K}$.

There is no need for the changes in height to be simultaneous for all the portions of air, and in practice it is much more convenient to let them be successive.

Varieties of particles.—I have observed the scattering of smoke from a smouldering wick, from burning weeds, from factory chimneys and from ship's funnels: also the scattering of portions of cloud near the horizon and of puffs of ammonium chloride from a special apparatus. Chimney smoke is not to be recommended, as it rises through the air. Clouds and steam may mislead one by

* *Vide* 'Computer's Handbook,' M.O. 223, Section V.

evaporating. The cold NH_4CL smoke proved more satisfactory in these ways. Lycopodium dust might be better still, as isolated grains would fall at a definite rate relative to the air. But I have not succeeded in making the lumps break up into grains. The downy parachute which carries the seed of the dandelion, *Taraxacum officinale*, has been found to be convenient. When the seed is broken off, the parachute falls at a rate of 10 to 15 cm. sec.$^{-1}$ in still air. The standard deviation of this rate must be allowed for. The formulæ for correction are given below. A brown parachute, twice as large each way as that of taraxacum, grows near Benson. I am indebted to the Botanical Department of the British Museum for a search among their dried specimens for a large white parachute. A splendid one came from an African plant called strophanthus. The parachute is about 6 cm. in diameter and has a long stalk by which it can be held conveniently. When the seeds were broken off the parachutes fell, in still air, at an average rate of 20 cm. sec.$^{-1}$.

Other artificial clouds, which have been used with success, are paraffin-oil vapour from an extinguished blast lamp, and smoke of phosphorus pentoxide made by dropping calcium phosphide into dilute hydrochloric acid. In strong winds the smoke from a firework known as "Vesuvius"* is convenient.

If one could mark and follow individual molecules equation (5) would give the molecular diffusivity in still air, 0·2 cm.^{2} sec.$^{-1}$. Actually what we observe is the centre of a small puff of smoke, and this is not constantly the position of the same molecules, so that in still air we find $K = 0$. To be perfectly exact all observations of K by this method should be increased by 0·2 cm.^{2} sec.$^{-1}$, an entirely negligible correction. In any theory the diffusivity depends on the motions which the theory does not follow in detail. In laboratory experiments, in which the molecular motion only is ignored, K is taken as 0·2 cm.^{2} sec.$^{-1}$. In meteorological telegraphy variations of wind of less than 10 minutes' duration are ordinarily ignored, and there is an appropriate, much larger, value of the diffusivity. In a certain scheme for numerical prediction it is proposed to average the wind over periods of 6 hours, and the further variations thus omitted must be taken into account by further increase in K. It follows that the puffs of smoke should be so small as to allow the smallest eddies to be observed, and, for the last-named purpose, that the observations should be spread over a period of 6 hours. In obtaining the data in the following table I believe the former condition has been fulfilled, but the latter has not. When only the order of K or ξ is required, it is enough to assume that the standard deviation is $\frac{1}{5}$ of the distance between the extremes of height observed, when the number of observations is about 40.

Observations very near the earth's surface have peculiarities. It is obvious that $\partial\chi/\partial h = 0$ at an impermeable horizontal surface. This condition can be satisfied in the integral by taking the portion of the distribution which would be cut off by the surface, reflecting it in the surface, and adding it to the rest of the distribution.

* Made by Messrs. C. T. Brock & Co.

The standard deviation is not then sufficient to give K. It is necessary to make more elaborate computations with K. PEARSON'S "incomplete normal moment functions."

These difficulties are avoided in the symmetrical case when the source of smoke is exactly on the ground, and the smoke does not rise or fall by its temperature.

The results of these measurements are set out in the last column of Table IV. It is seen there that K increases from 5 near the surface of land up to 10,000 at the height of a factory chimney.

But before considering the results further a fuller mathematical investigation will now be made.

V. GENERAL THEORY OF EDDY-DIFFUSIVITY DEDUCED FROM SCATTERING.

The foregoing theory of the diffusion of a lamina assumes that the diffusivity is constant throughout the space, and that the density in the lamina does not vary except in the smooth regular manner indicated by the "law of error." But it is well known that the wind has an intricate structure. Thus if observations of the smoke puffs are to yield a measure of the diffusivity from the formula

$$\text{diffusivity} = \frac{\text{increase in square of standard deviation}}{\text{twice corresponding increase in time}},$$

then either the interval of time in the denominator must be long compared with the fluctuations of the wind in time, or the initial standard deviation must be large compared with the fluctuations of the wind in space, or both conditions must hold. The former condition is an inconvenient one in practice, because puffs are apt to fade before a sufficient time has passed. Dandelion parachutes, with the seeds removed, may be better than smoke for this purpose.

The following theory brings to light some of the assumptions involved in the measurement of diffusivity by smoke puffs. It was contrived specially in order to avoid "the distance through which an eddy moves before mixing with its surroundings," a quantity which occurs in TAYLOR'S theory, but which does not lend itself easily to measurement, except in the case of cumulus eddies. See Section IX. below.

The potential* temperature Θ does not change at a point moving with fluid, if radiation and precipitation can be neglected. Now let a portion of an eddy move from a height h_1 at time t_1 to a height h_2 at t_2. Then, regarding Θ as a function of h and t, we have

$$\Theta(h_1, t_1) = \Theta(h_2, t_2). \qquad (1)$$

* Potential temperature is the temperature which the air would acquire if compressed adiabatically to a standard pressure. If Θ is to be of service in dealing with cloudy air the standard pressure must be high enough to evaporate the cloud in all samples.

From each side of (1) subtract $\Theta(h_1, t_2)$ and divide through by t_2-t_1. Then

$$\frac{\Theta(h_1, t_1)-\Theta(h_1, t_2)}{t_2-t_1} = \frac{\Theta(h_2, t_2)-\Theta(h_1, t_2)}{t_2-t_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now the left side of (2) is the finite difference ratio $\frac{\delta\Theta}{\delta t}$ at the height h_1, which is what we want to find in terms of the spacial distribution of Θ. Expand the right-hand side of (2) in powers of h_2-h_1 by the well-known theorem in the calculus. It follows that, if subscripts indicate the time and height

$$\left(\frac{\delta\Theta}{\delta t}\right)_{h_1, \frac{t_2+t_1}{2}} = \frac{1}{t_2-t_1}\left\{\left(\frac{\partial\Theta}{\partial h}\right)_{t_2 h_1}(h_2-h_1)+\left(\frac{\partial^2\Theta}{\partial h^2}\right)_{t_2 h_1}\frac{(h_2-h_1)^2}{2\,!}+\text{higher terms.}\right. \quad . \quad . \quad (3)$$

The difference ratio on the left of this equation is centred at the same height h_1 as the differential coefficients on the right of the same, but at a time $\frac{1}{2}(t_2-t_1)$ previous. This slight misfit in centering will not matter, because t_2-t_1 will be of the order of one minute or less, whereas we are next going to take the average of each term in (3) over a much longer time, say 6 hours. The subscripts may now be omitted as unnecessary. Let a bar over a symbol, or group of symbols, denote the mean value over this longer period. Let a dash denote the instantaneous deviation from this mean, so that, for instance, we have for every fluctuating quantity a formula such as

$$\frac{\partial\Theta}{\partial h} = \overline{\left(\frac{\partial\Theta}{\partial h}\right)}+\left(\frac{\partial\Theta}{\partial h}\right)' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now the mean of any dashed quantity vanishes. . . . . (5)

Again the mean of the product of any dashed quantity into any barred quantity also vanishes. . . . . (6)

We shall further suppose that the mean of h_2-h_1 vanishes, ... (7)

that is to say that there is no mean vertical displacement.

Then, in the first term on the right of (3)

$$\begin{aligned}\overline{\frac{\partial\Theta}{\partial h}(h_2-h_1)} &= \overline{\left\{\overline{\left(\frac{\partial\Theta}{\partial h}\right)}+\left(\frac{\partial\Theta}{\partial h}\right)'\right\}\left\{\overline{(h_2-h_1)}+(h_2-h_1)'\right\}} \\ &= \overline{\left(\frac{\partial\Theta}{\partial h}\right)'(h_2-h_1)'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)\end{aligned}$$

because of (6) and (7).

Now $\left(\frac{\partial\Theta}{\partial h}\right)'(h_2-h_1)'$, after being divided by t_2-t_1 and by the "standard deviations" of $\left(\frac{\partial\Theta}{\partial h}\right)'$ and of $(h_2-h_1)'$, becomes equal to the correlation between $\partial\Theta/\partial h$ and

$(h_2-h_1)/(t_2-t_1)$. So the first order term on the right of (3) vanishes, on taking the mean, if the variations of $\partial\Theta/\partial h$, in time at a fixed point, are not correlated with the variations in $(h_2-h_1)/(t_2-t_1)$, at the same point and time.

In cumulus cloud eddies, the variations of velocity are caused by variations of the potential temperature Θ, so that a correlation is almost certain to exist. On the contrary, when the eddies are due to dynamical instability, the correlation may be expected to vanish. In the latter case, it is the second order term of the right of (3) which becomes effective, so that

$$\overline{\frac{\delta\Theta}{\delta t}} = \overline{\frac{\partial^2\Theta}{\partial h^2} \cdot \frac{(h_2-h_1)^2}{2(t_2-t_1)}} \qquad (9)$$

Now suppose further that either $\partial^2\Theta/\partial h^2$ has no variations at a fixed time and level, or else that its variations are not correlated with those of $(h_2-h_1)^2$. Then (9) simplifies to

$$\overline{\frac{\delta\Theta}{\delta t}} = \overline{\frac{\partial^2\Theta}{\partial h^2}} \cdot \frac{\overline{\{(h_2-h_1)^2\}}}{2(t_2-t_1)} \qquad (10)$$

Thus

$$\overline{(h_2-h_1)^2}/2(t_2-t_1) \text{ is the eddy-diffusivity K} \qquad (11)$$

It is seen to be identical with that derived in Part IV. above, by considering the diffusion of a lamina, in which the density was distributed according to the law of error. It is a quantity easily measured.

Of course if t_2-t_1 were sufficiently small, say $\frac{1}{10}$ second, then it would be the first power of h_2-h_1, which would be proportional to t_2-t_1, instead of the square. This suggests that t_2-t_1 must be long compared with the fluctuations of the wind. On the other hand t_2-t_1 must be short compared with the period, of say 6 hours, over which the averages denoted by the bar are desired to be taken.

A similar argument can be applied to any other quantity which, like Θ, does not change following the motion of the fluid, provided it has space-rates independent of the time-variations of velocity. Thus the mass-of-water-per-unit-mass-of-atmosphere may replace Θ in (10) with similar restrictions.

When we consider diffusion in three dimensions there may be six coefficients of diffusivity corresponding to the six components of stress.

VI.—Osborne Reynolds' Eddy-stresses.

But we cannot, without further investigation, apply the preceding argument to the diffusion of horizontal velocity in a fixed azimuth.

Something might perhaps be deduced from the well-known theorem that, when ρ is constant,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\tfrac{1}{2}\rho v^2+\rho\psi+p\right) = \frac{\partial p}{\partial t} \qquad (12)$$

where ψ is the gravity potential.

The eddy-viscosity can however be measured rigorously by smoke-puff observations made in such a manner as to fit in with OSBORNE REYNOLDS' theory of eddy-stresses.* This theory is remarkably free from assumptions which might limit its generality. It is to be found in LAMB's 'Hydrodynamics,' IV. edn., Art. 369.

The equations of motion are three, such as

$$-\frac{\partial(\rho v_X)}{\partial t} = \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\rho v_X v_X) + \frac{\partial}{\partial y}(\rho v_X v_Y) + \frac{\partial}{\partial h}(\rho v_X v_H) + \rho\frac{\partial\psi}{\partial x} - 2\omega\sin\phi v_Y - c\left(\tfrac{1}{3}\frac{\partial\,\mathrm{div}\,v}{\partial x} + \nabla^2 v_X\right) \quad . \quad . \quad . \quad . \quad (13)$$

where c is the "molecular" or "ordinary" viscosity. Note that there is no need to assume ρ to be independent of position. REYNOLDS assumed this, but for a reason that does not concern us. It will be necessary however to assume that ρ', the variation of density at a fixed point, is so much smaller in comparison with $\bar{\rho}$ than is v' in comparison with $\bar{v}$, that we may put $\rho' = 0$. This being so, we find on taking the mean that (13) becomes

$$-\overline{\left\{\frac{\partial(\rho v_X)}{\partial t}\right\}} = \frac{\partial\bar{p}}{\partial x} + \rho\frac{\partial\psi}{\partial x} - 2\omega\sin\phi\bar{v}_Y - c\left(\tfrac{1}{3}\frac{\partial\,\mathrm{div}\,\bar{v}}{\partial x} + \nabla^2\bar{v}_X\right) + \frac{\partial}{\partial x}(\rho\bar{v}_X\,.\,\bar{v}_X + \rho\overline{v'_X\,.\,v'_X}) + \frac{\partial}{\partial y}(\rho\bar{v}_X\,.\,\bar{v}_Y + \rho\overline{v'_X\,.\,v'_Y}) + \frac{\partial}{\partial h}(\rho\bar{v}_X\,.\,\bar{v}_H + \rho\overline{v'_X v'_H}) \quad . \quad . \quad . \quad (14)$$

The left side of (14) is the difference between ρv_X at the beginning and at the end of the period through which the average is taken, divided by the period; and that is what we want. The right side of (14) is of exactly the same form in the mean quantities $\bar{p}$, $\bar{v}_X$, $\bar{v}_Y$, $\bar{v}_H$ as (13) was in the corresponding instantaneous quantities p, v_X, v_Y, v_H; except that there is added a force per unit volume in the x direction equal to minus

$$\frac{\partial}{\partial x}(\rho\overline{v'_X v'_X}) + \frac{\partial}{\partial y}(\rho\overline{v'_X v'_Y}) + \frac{\partial}{\partial h}(\rho\overline{v'_X v'_H}). \quad . \quad . \quad . \quad (15)$$

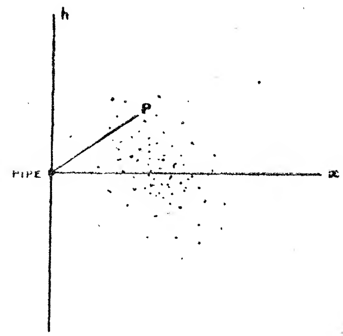
On working out the corresponding equations for the y and h components, it is seen that this additional force per unit volume is just that which would be given by the following systems of stresses

$$\left.\begin{array}{lll} \widehat{xx} = -\rho\overline{v'_X v'_X}\,; & \widehat{yy} = -\rho\overline{v'_Y v'_Y}\,; & \widehat{hh} = -\rho\overline{v'_H v'_H} \\ \widehat{xy} = -\rho\overline{v'_X v'_Y}\,; & \widehat{yh} = -\rho\overline{v'_Y v'_H}\,; & \widehat{hx} = -\rho\overline{v'_H v'_X} \end{array}\right\} \quad . \quad . \quad . \quad (16)$$

* Major G. I. TAYLOR tells me that he attempted to measure $\widehat{xh}$ with a balloon on an elastic tether in 1914.

when any symbol such as $\widehat{xy}$ is the force in the x-direction per unit area of a plane normal to the y axis. Tractions are reckoned positive, as usual. The above is taken from OSBORNE REYNOLDS' theory, adapted and slightly generalized to suit our needs for a rotating atmosphere, having density diminishing with height and a molecular viscosity which is not neglected.

One may form a clear mental picture of these eddy-stresses by imagining the scattering of smoke puffs. Let a puff emerge from a pipe at the origin of the co-ordinates. After a short interval of time τ, let the puff appear at the point P on the diagram, as seen by an observer at a distant point on the y-axis. Now let the observation be repeated for a large number of puffs in succession, the time τ being kept the same for each. We thus obtain a diagram, with a large number of points on it, showing the scattering of the puffs after τ. Then the eddy-stresses are simply related to the correlations and standard deviations of this scatter-diagram—under certain conditions. For let X, Y, H now mean the co-ordinates of any one of the dots on the diagram reckoned from the source of smoke.



Then the velocities of the corresponding puff were

$$v_X = \frac{X}{\tau}; \qquad v_H = \frac{H}{\tau}, \quad \ldots \ldots \ldots \quad (17)$$

provided the time τ was so short that, during it, the velocity may be regarded as uniform and in a straight line. (18)

Again, the velocity of the puff will be equal to that of the air which it has replaced provided the puff is at the same temperature as the air, and provided that the pipe points parallel to the Y-axis so that the impulse with which the puff leaves the pipe does not show in the projection on the plane XOH.

Let us suppose that a number of puffs, n in all, are observed. In order to correspond with the time-mean taken over 6 hours, which was used in deriving the eddy-stresses from the equations of motion, these n puffs should be spread uniformly over a similar interval.

From the scatter diagram we can compute first the mean velocities. For the mean velocities are

$$\bar{v}_X = \frac{\overline{X}}{\tau} = \frac{1}{\tau \, . \, n} \Sigma X; \qquad \bar{v}_Y = \overline{Y}/\tau = \frac{\Sigma Y}{\tau \, . \, n}, \quad \ldots \ldots \quad (19)$$

where Σ has the meaning :—take the sum of what follows it, for n puffs.

Then the normal eddy-stresses, $\widehat{xx}$ and $\widehat{hh}$, are found thus

$$\widehat{xx} = -\rho\overline{v'_{\mathrm{X}}v'_{\mathrm{X}}} = -\rho\,\overline{(v_{\mathrm{X}}-\bar{v}_{\mathrm{X}})\,(v_{\mathrm{X}}-\bar{v}_{\mathrm{X}})}$$

$$= -\frac{\rho}{\tau^2 n}\Sigma\,(\mathrm{X}-\overline{\mathrm{X}})\,(\mathrm{X}-\overline{\mathrm{X}}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

So

$$\widehat{xx} = -\rho\sigma_{\mathrm{X}}^2/\tau^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where σ_{X} is the "standard deviation" of the dots in the x-direction. If σ_{H} is the corresponding quantity vertically

$$\widehat{hh} = -\rho\sigma_{\mathrm{H}}^2\tau^{-2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

So the direct eddy-stress in the direction of the wind is intimately related to the gustiness shown by a tube-anemometer.

The shearing eddy-stress

$$\widehat{xh} = -\rho\overline{v'_{\mathrm{X}}v'_{\mathrm{H}}} = -\frac{\rho}{\tau^2}\frac{1}{n}\,\Sigma\,(\mathrm{X}-\overline{\mathrm{X}})\,(\mathrm{H}-\overline{\mathrm{H}}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

So

$$\widehat{xh} = -\rho\tau^{-2}r_{\mathrm{XH}}\,\sigma_{\mathrm{X}}\sigma_{\mathrm{H}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where r_{XH} is the correlation between the co-ordinates X and H of the dots.

By projecting the puffs on the other two co-ordinate planes we should be able to measure similarly the remaining components of eddy-stress.

To find the eddy-viscosity we must compare the sheering eddy-stress $\widehat{xh}$ with $\left(\frac{\partial\bar{v}_{\mathrm{X}}}{\partial h}+\frac{\partial\bar{v}_{\mathrm{H}}}{\partial x}\right)$, which is the rate of shearing strain in the *mean* motion. Usually $\partial\bar{v}_{\mathrm{H}}/\partial x$ is negligible, so that the rate of shearing is $\partial\bar{v}_{\mathrm{X}}/\partial h$, a quantity which can easily be observed. At first sight one might think that $\partial\bar{v}_{\mathrm{X}}/\partial h$ was simply related to the slope of the regression line in the scatter diagram; but on examination this proves not to be the case. The slope of the regression line is independent of τ, because (18) is satisfied for all permissible intervals of time.

It should be noted that no shearing stress such as $-\rho\overline{v'_{\mathrm{X}}v'_{\mathrm{H}}}$ can exceed, in absolute value, the geometric mean of the corresponding pair of direct stresses $-\rho\overline{v'_{\mathrm{X}}v'_{\mathrm{X}}}$, $-\rho\overline{v'_{\mathrm{H}}v'_{\mathrm{H}}}$ for the same reason that a correlation coefficient cannot exceed unity.

The probable errors of eddy-stresses, determined from the scattering of particles moving with the air, may be taken to be as follows

$$\text{Probable error of } \widehat{xx} = 0{\cdot}674\,.\,\widehat{xx}\sqrt{\frac{2}{\mathrm{N}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\text{Probable error of } \widehat{xh} = 0{\cdot}674\sqrt{\frac{\widehat{xh}^2+\widehat{xx}\,.\,\widehat{hh}}{\mathrm{N}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

These are based on the assumption that the number of particles is not less than say 20, and that the scatter is "normal," so that the logarithm of the density of particles would be a quadratic function of x, y, h if a very large number of particles were observed. Reference may be made to the fundamental papers on probable errors: FILON and PEARSON, 'Phil. Trans.,' A, vol. 191; W. F. SHEPPARD 'Phil Trans.,' A, vol. 192. Equation (26) follows from the probable error of the quantity called by K. PEARSON the "product moment coefficient *taken about the mean.*" Equation (25) follows from (26) on putting $x = h$, or may be deduced independently from the probable error of a standard deviation taken about the mean.

Corrections for the Motion of the Parachute Relative to the Air.

When observing eddy-stresses by the aid of the parachutes of plant seeds it is desirable to allow for the velocity of the parachute in still air. For large specimens of the parachute of *Taraxacum officinale*, after cutting off the seed, the velocity in still air was found to have a mean of 12 cm. sec.$^{-1}$ with a standard deviation of 2 cm. sec.$^{-1}$. It may be shown from the equation of motion that this limiting velocity is acquired in a negligibly short interval of time. Thus call the limiting velocity c downwards, the instantaneous velocity downwards u. Then if the friction is proportional to the velocity

$$g\,(\text{mass}) = c \times \mathrm{F}, \qquad (27)$$

where F is a constant. But, when accelerating,

$$(\text{mass})\,.\,\frac{du}{dt} = g\,(\text{mass}) - u\mathrm{F}. \qquad (28)$$

Eliminate the mass between these two equations and there results

$$\frac{c}{g}\,\frac{d\,(u-c)}{dt} = -\,(u-c). \qquad (29)$$

So that the discrepancy between the actual velocity u and the terminal velocity c sinks to e^{-1} of itself in a time equal to c/g, which for the taraxacum parachute having $c = 12$ cm. sec.$^{-1}$ would be only a hundredth of a second.

Less negligible is the variation of the velocity c from one parachute to another. What we actually observe is not v_{H} the upward velocity of the air, but $v_{\mathrm{H}} - c$. Now write $\bar{c}$ for the mean velocity of the parachute in still air, and c' for the deviation from the mean. Then in finding the direct stress $\widehat{hh}$ we must perforce work out first the "raw" moment $\{\overline{(v_{\mathrm{H}}-c)^2}\} = \overline{[\{(\bar{v}_{\mathrm{H}}-\bar{c})+(v'_{\mathrm{H}}-c')\}^2]}$. On expanding and remembering that a bar put over the product of a dashed and a barred symbol, causes the result to vanish, and also that $\overline{v'_{\mathrm{H}}-c'} = 0$, it is found that

$$\{\overline{(v_{\mathrm{H}}-c)^2}\} = (\bar{v}_{\mathrm{H}}-\bar{c})^2 + \overline{v'_{\mathrm{H}}v'_{\mathrm{H}}} + \overline{c'c'}$$

it follows that the corrected stress is given by

$$\widehat{hh} = -\rho\overline{v'_{\mathrm{H}}v'_{\mathrm{H}}} = -\rho\,\{\overline{(v_{\mathrm{H}}-c)\,(v_{\mathrm{H}}-c)}-(\bar{v}_{\mathrm{H}}-\bar{c})^2-\overline{c'c'}\} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

which is the actual formula used in working up the observations. Here $\overline{c'c'}$ is the square of the standard deviation of the velocity of the parachutes in still air.

Next we require the product moments in order to find the shearing stresses $\widehat{xh}$ and $\widehat{yh}$. The raw moment $\overline{v_{\mathrm{X}}\,(v_{\mathrm{H}}-c)}$ is found on expanding to be equal to $\bar{v}_{\mathrm{X}}\,\overline{(v_{\mathrm{H}}-c)}+\overline{v'_{\mathrm{X}}v'_{\mathrm{H}}}$.

So that the corrected value for the stress is

$$\widehat{xh} = -\rho\overline{v'_{\mathrm{H}}v'_{\mathrm{X}}} = -\rho\,\{\overline{v_{\mathrm{X}}\,(v_{\mathrm{H}}-c)}-\bar{v}_{\mathrm{X}}\,\overline{(v_{\mathrm{H}}-c)}\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and as $\overline{c'c'}$ does not appear, the scatter of the velocities of the parachutes in still air does not make a correction necessary for the shearing stresses.

VII. Summary of Theory of Scattering of Particles of Air.

The conclusion we have reached is the following. For any sort of eddy, whether due to "dynamical instability," or to the rising of heated air in cumuli, the eddy-stresses are best measured by equations (22), (24) and the like, because the theory from which they are derived is very general; and the eddy-viscosity is best measured as the ratio of the shearing eddy-stress to the rate of mean shearing strain. It is conceivable that $\widehat{xh}$ found from (24) might turn out to be zero. In that case it would be necessary to investigate effects of higher order. This might possibly be done by developing, for the quantity $(\frac{1}{2}\rho v^2+\rho\psi+p)$ in equation (12) an analysis similar to that of (1) to (11) for potential temperature. The diffusivity for potential temperature, on the other hand, should be measured differently according as the eddies are produced by variations of potential temperature or not. Thus for cumulus eddies we should take the mean of (3), retain the linear term on its right-hand side and neglect the quadratic one. Then t_2-t_1 must be small, so that

$$\overline{\frac{\delta\Theta}{\delta t}} = \overline{\frac{\partial\Theta}{\partial h}\cdot\frac{(h_2-h_1)}{t_2-t_1}} = \overline{\left(\frac{\partial\Theta}{\partial h}\right)'v'_{\mathrm{H}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31\mathrm{A})$$

The diffusivity is measured as the right side of this equation divided by $\overline{\partial^2\Theta/\partial h^2}$. Thus

$$\mathrm{K} = \frac{\overline{\left(\frac{\partial\Theta}{\partial h}\right)'v'_{\mathrm{H}}}}{\overline{\partial^2\Theta/\partial h^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

But for eddies due to dynamical instability, neglect the linear term in (3) and measure the diffusivity as

$$\mathrm{K} = \frac{(h_2-h_1)^2}{2\,(t_2-t_1)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where (t_2-t_1) must not be too small. It will be interesting to see whether eddy-diffusivity is found to be equal to eddy-viscosity divided by density.

Let us now see how these two modes of disposal of smoke fit together—the one, at short intervals of time, governed by the eddy-stresses, the other at long intervals governed by the diffusivity. Let us draw the trail of smoke as it would appear to a distant observer looking, say, horizontally. The scatter may be represented by drawing two lines through the points where the smoke-density has its standard deviation in height above or below its mean. Let T be the time since the smoke emerged from its source. Suppose that the stress $\widehat{hh}$, the diffusivity K and the mean velocity $\bar{v}_X$ are all constant along the path of the smoke. The time T taken to travel a horizontal distance x measured down the trail from the source is $x/\bar{v}_X$. Near the origin $T = \tau$ in equation (22), and so the standard deviation in height is

$$\pm \frac{x}{\bar{v}_X} \sqrt{-\widehat{hh}/\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

representing a pair of straight lines intersecting at the source. Further down the trail $T = t_2 - t_1$ in equation (11), and the standard deviation in height is

$$\pm \sqrt{x} \cdot \sqrt{\frac{2K}{\bar{v}_X}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

representing a parabola with horizontal axis and its apex at the source. Thus according to the theory, the smoke may be said to be contained within a paraboloid which has had its blunt end sharpened into a cone. The preceding theory gives us no clue as to the manner of transition from the cone to the paraboloid.

When we can observe a sufficient length of *the path traced out in space by a single small portion of air* the eddy-stresses and the eddy-diffusivity may be deduced from the irregularities in the motion. With this object I have observed the motion of anti-aircraft shell bursts, and of portions of cloud, by means of an Abney level or a pocket sextant. With better instruments this method might yield a good deal of information about eddies at heights such as 2 to 5 km. One principal difficulty is that the shell-burst fades away after about 5 minutes, before a sufficient length of path has been observed to give the diffusivity.

If the path is sufficiently high and long, the hills, trees and houses on the earth may be regarded as blending into a "roughness." Suppose this roughness to be uniform. Then if we had been causing smoke to issue in puffs from a fixed pipe, we should presumably have obtained the same scatter diagram for the puffs, within the limits of probable error, at whatever point of the path we had placed the pipe, or at whatever time we had begun to observe, within limits. If this is so, we may form the scatter diagram by taking its origin at every point in succession of the trajectory of the single particle. For instance, Captain CAVE in his book on "The Structure of the Atmosphere," gives several diagrams of the irregularities of height of a balloon observed by two theodolites, the uniform vertical motion of the balloon relative to the air having been eliminated. From his figure 30, of an ascent on February 19, 1909, the following has been deduced, by taking the origin of scattering at every available

minute mark, and working out the standard deviation, in height about the mean height, after a time α.

TABLE III.

α, secs.	1×60	3×60	5×60	7×60	9×60
$\frac{\sigma_H^2}{2\alpha} = 10^4 \times$ cm.^{2} sec.$^{-1}$	0·93	0·82	0·55	0·55	0·33
$\rho\frac{\sigma_H^2}{\alpha^2}$ dyne cm.$^{-2}$	0·345	0·100	0·041	0·029	0·014
Number of points	14	12	10	8	6

Now $\sigma_H^2/2\alpha$ would be the diffusivity if α were a "long" time; and the criterion of sufficiency in length is that $\sigma_H^2/2\alpha$ should not vary with α. The small number of points makes the probable errors large, so that the decrease of $\sigma_H^2/2\alpha$ between $\alpha = 7$ mins. and $\alpha = 9$ mins. is not significant. It looks as though the diffusivity K were here of the order of $0{\cdot}5 \times 10^4$ cm.^{2} sec.$^{-1}$.

Again $-\rho\sigma_H^2/\alpha^2$ would be the eddy-stress, $\widehat{hh}$, if α were so small that further decrease made no further change in the quantity. This stage is not reached at $\alpha = 1$ minute. All that we can say is that the eddy-stress is probably greater than 0·345 dynes cm.$^{-2}$.

The mean height of this observation is 1 km. above ground, and the mean velocity 1270 cm. sec.$^{-1}$.

The photograph of a smoke trail in fig. 2 suggests that the eddying is partly random but also partly sinusoidal. Let us therefore see what would happen *if the path of the particle were an exact sine curve* without any random variations. Let the height h of the particle be given by $h - \mathrm{B} = \mathrm{A} \sin qt$ where A, B and q are constants. The increase in height in a time α would be

$$\mathrm{A}\left[\sin\left\{q\left(t+\frac{\alpha}{2}\right)\right\} - \sin\left\{q\left(t-\frac{\alpha}{2}\right)\right\}\right] = \mathrm{A} \,.\, 2\cos qt \,.\, \sin\left(q\frac{\alpha}{2}\right)$$

by trigonometry. So that the standard deviation σ_H after a time α would be given by

$$\sigma_H^2 = \frac{4\mathrm{A}^2\left(\sin\frac{q\alpha}{2}\right)^2}{\mathrm{L}}\int_{t=0}^{t=\mathrm{L}} (\cos qt)^2\, dt$$

where L is a very long time. The integral is equal to $\frac{1}{2}$L plus a negligible oscillatory part. Consequently $\sigma_H^2 = 2\mathrm{A}^2\left(\sin\frac{q\alpha}{2}\right)^2$. It follows that the stress $\widehat{hh}$, which is the limit of $-\rho\sigma_H^2/\alpha^2$ when α is small, comes to $-\rho q^2\mathrm{A}^2/2$; whereas the diffusivity K, which is the limit of $\sigma_H^2/2\alpha$ when α is long, comes to zero.

Comparison with TAYLOR'S *expression for the Diffusivity and with* W. SCHMIDT'S *"Austausch."*

In TAYLOR'S remarkable investigation ('Phil. Trans.,' A, vol. 215) from which the present research took its stimulus, the diffusivity K is given, in the present notation, as the mean value of $v_H (h-h_0)$ over a large horizontal plane; and it is stated that $h-h_0$ is the height through which an eddy moves from the layer at which it was at the same temperature as its surroundings, to the layer with which it mixes. This definition of $h-h_0$ is puzzling, for it seems impossible to reconcile the supposed starting and stopping of the air, with the ceaseless motion which we observe in nature, except in the case of cumulus eddies. Happily we are now in a position to clear away the mystery. For it has been shown independently in the present paper that the diffusivity is given by

$$K = \overline{(h_2-h_1)^2/2(t_2-t_1)}$$

where (t_2-t_1) is a time long compared to the fluctuations in the wind, and where the bar implies an average taken over a still much longer time. As (t_2-t_1) is the same for all the quantities which are averaged, we may remove it from under the bar, writing

$$K(t_2-t_1) = \tfrac{1}{2}\overline{(h_2-h_1)^2}.$$

Differentiate this equation with respect to t_2,

$$K = \overline{(h_2-h_1)\frac{d}{dt_2}(h_2-h_1)} = \overline{(h_2-h_1) \,.\, v_H} \text{ at } t_2,$$

thus K is expressed as the mean of the product of the rise in height during a long time into the vertical velocity *at the end* of that time. It may also be taken at the beginning. Comparing with TAYLOR'S form quoted above we see a strong resemblance, and we are led to suppose that TAYLOR'S theory makes two unnecessary and unnatural restrictions: (1) that the portion of air should start at the same temperature as its surroundings; (2) that the portion of air should finally mix with its surroundings. But that if these restrictions be removed, then another becomes necessary, namely that (t_2-t_1) should be sufficiently long (several minutes). Whether the average be taken over a large horizontal plane, or over a very long time (6 hours), appears to be a matter of indifference.

The extent to which TAYLOR assumes viscosity to be independent of height in his *general* theory ('Phil. Trans.,' A, vol. 215, pp. 11 to 13) is this: he neglects the terms due to the initial eddying in his equation (6). That is a doubtful proceeding, unless the initial eddying is zero: but zero is independent of height.

The "Austausch" of W. SCHMIDT is defined by him (in 'Sitz. Akad. Wiss.,' Wien (1917), pp. 4 to 5) as

$$\frac{\Sigma\,(\text{element of mass crossing horizontal plane}) \times (\text{vertical displacement of element})}{(\text{whole area}) \times (\text{time of motion})}$$

If we replace the summation by an integration over a large area S in the plane of xy, the element of mass crossing per unit time is $dx\, dy\, \rho v_{\rm H}$, so that the "Austausch" becomes $\frac{1}{\rm S}\iint \rho v_{\rm H}\,(h-h_0)\, dx\, dy$ which is ρ times TAYLOR'S diffusivity as defined in 'Phil. Trans.,' A, vol. 215, p. 3. Thus we must suppose that W. SCHMIDT'S definition of "Austausch" requires amplification concerning the interval of time and concerning the position in it of the velocity, just as TAYLOR'S definition of K does.

VIII. NUMERICAL VALUES DERIVED FROM THE SCATTERING OF PARTICLES.

Fig. 1 is a photograph* of the trail of paraffin vapour from an extinguished blast-lamp which projected the vapour in a direction at right angles to the wind. It shows a cone, with a blunt point due to the finite size of the source of smoke, passing smoothly into a form, which certainly diverges less rapidly than the initial cone, and which looks like a paraboloid. Opinion might differ slightly as to where to draw the lines corresponding to the standard deviation of smoke. In a "normal" distribution 0·68 of the whole number of particles lie between the two standard deviations. If the lines are placed as in the accompanying black and white drawing, then it follows, as the mean velocity of the smoke was 1·7 metres/sec., and the density of the air was $1·21 \times 10^{-3}$ grm. cm.$^{-3}$, that

$$\text{stress}\ \widehat{hh} = -0·73 \text{ dyne cm.}^{-2} \quad \text{diffusivity} = \mathrm{K} = 240 \text{ cm.}^2 \text{ sec.}^{-1}$$
$$\text{turbulivity} = \xi = 340 \text{ grm.}^2 \text{ cm.}^{-2} \text{ sec.}^{-5}.$$

This photograph was taken in the evening, when the day-wind was diminishing. The source was 190 cm. above ground. Obstructions to windward only subtended an angle of 2°·1 at the source of smoke. The exposure lasted 60 seconds.

Fig. 2 was taken five minutes later in the same place, with an exposure of 85 seconds.

The velocity of the smoke had decreased to 1·3 metres per sec. The measurements yield

$$\widehat{hh} = -1·2 \text{ dynes cm.}^{-2}; \qquad \mathrm{K} = 750 \text{ cm.}^2 \text{ sec.}^{-1}; \qquad \xi = 1050 \text{ grm.}^2 \text{ cm.}^{-2} \text{ sec.}^{-5}.$$

In this case the photograph shows a distinct neck between the cone and the paraboloid, at a distance from the source roughly 1·3 times its height above ground. This neck can also be recognized in some other photographs. Its presence signifies that the motion of the air was compounded of (i.) a random eddying, plus (ii.) a wave motion in which the particle of air executed a wave having a length, relative to a point fixed to the earth, roughly 2·6 times the height of the particle above ground.

* Taken at Benson.

TABLE IV.—Observed Values of

The co-ordinate axes are taken to point: $0x$ horizontally with the mean wind at quantities are in C.G.S. units. Tractions are reckoned positive, so that $\widehat{xh}$ and $\widehat{yh}$

Entropy gradient vertically upwards.	Surface.	Methods.	Notes.	h cm. above surface.	$\bar{v}_x$ $\frac{\text{cm.}}{\text{sec.}}$	$\widehat{xx}$ dynes per square centi-metre.
Positive sunrise . .	Tall grass . . .	Smoking wick . . .	I.	30 170 300,000	10 60 200	—
Small ?	Oat field . . .	NH_4Cl puffs . . . . NH_4Cl puffs . . . . Dandelion down . .	II.	40 100 50	20 40 —	— — —
Small ?	Corn field . . .	Smoke puffs	III.	120	100	$-0 \cdot 1$
Small ?	Moor	Smoking wick . . .	IV.	165	100	—
?	Moor, trees . .	Thistle down . . .	V.	200	145	$-2 \cdot 4$ $\pm 0 \cdot 5$
Positive ?	Flat field . . .	Paraffin vapour . . .	VI. Figs. 1 and 2	190	170 140	— —
1919, Mar. 4 Between England and the Isle of Wight.	Sea	Steamer's smoke . .		4,000	100 ?	—
1919, Apr. 7, 18h.	Sea	Steamer's smoke . .		10,000	200 ?	—
1919, July 4, 11h. over-cast	Sea	Steamer's smoke . .		2,000	130	—
1919, July 5, 12h.	Sea	Steamer's smoke . .		1,400	140	—
	Ditcham . . .	Capt. CAVE'S balloon .	p. 16	100,000	1270	—
	Flat fields . . .	Phosphorus pentoxide	VI. Fig. 3	340	120	—

Eddy-stresses and of *Eddy-diffusivity*.

the level of observation, $0y$ horizontally to the left and $0h$ vertically upwards. All are positive when the air above drags the air below in senses of x and y increasing.

$\widehat{yy}$ dynes per square centi-metre.	$\widehat{hh}$ dynes per square centi-metre.	$\widehat{xy}$ dynes per square centi-metre.	$\widehat{yh}$ dynes per square centi-metre.	$\widehat{hx}$ dynes per square centi-metre.	$\dfrac{\partial \bar{v}_x}{\partial h}$ sec.$^{-1}$.	Conductivity $= \rho K$ grm. cm.$^{-1}$ sec.$^{-1}$.	Diffusivity. K cm.^{2} sec.$^{-1}$.	Turbulivity. $\xi = g^2\rho^2 K$ grm.^{2} cm.$^{-2}$ sec.$^{-5}$.
—	—	—	—	—	>0·5 — —	0·006 — —	5 — —	7 — —
— — —	−0·004 −0·006 —	— — —	— — —	— — —	$\left.\right\}$0·3 $\left\{\right.$ —	— — 0·07	— — 60	— — 80
—	−0·2	—	—	<0·1	<0·06			
—	−0·04	—	—	—	—	0·03	24	34
− 2·9 ± 0·6	−0·6 ±0·12	[+0·45] ±0·39	−0·48 ±0·20	[−0·34] ±0·18				
— —	−0·7 −1·2	—	—	—	—	0·3 0·9	240 750	340 1,050
—	—	—	—	—	—	12	10,000	14,000
—	—	—	—	—	—	0·25	200	300
—	—	—	—	—	—	6·8	5,500	8,000
—	—	—	—	—	—	5·9	4,800	7,000
—	< −0·34	—	—	—	—	6	5,000	6,000
—	—	—	—	—	—	0·16	130	200

TABLE IV.—Observed Values of

Entropy gradient vertically upwards.	Surface.	Methods.	Notes.	h cm. above surface.	$\bar{v}_x$ $\frac{\text{cm.}}{\text{sec.}}$	$\widehat{xx}$ dynes per square centimetre.
1919, Oct., 21d. 12h., BENSON	Obstructions up wind $\frac{1}{50}$ radian	Paraffin vapour . . .		160	250	—
1919, Oct., 29d. 15h., BENSON ; raining	Obstructions up wind 0·045 radian	Smoke		190	330	—
Small ?	Moor, trees . .	Burning rubbish . .	VII.	900	300	—
Small ?	Fields, trees . .	Factory chimney . .	VIII.	3,000	600	—
Negative ?	Moor, low hills .	Cloud	IX.	150,000 2,000	1000 200	— —
Small ?	Wooded hills . .	Large fire	X.	25,000 37,000	— 1600	—
		Shell-puffs	XI.	300,000	500	—
Positive ?	Spurn head . .	Dines anemometer .	XII.	—	1200	− 80
Positive ?	Open sea . . .	Steamer's smoke . .	XIII.	1,500	—	—
Negative ?	Open sea . . .	Steamer's smoke . .	XIV.	5,000	200	—
?	Open sea . . .	Alto-stratus	XV.	500,000	—	—

Notes to Table of Eddy-Stresses and Diffusivities.

I. 1917, June, 16d. 4h. 8m. L.A.T., hilltop near Ancemont, France. Standing hay composed of a species of *Festuca* (identified by my friend, Mr. SAM PIM). It grew fairly densely to 30 cm. from the ground and tall seed stems rose to 70 cm.

II. 1917, July, 16d. 19½h. L.A.T., Maffrecourt, France. Green oats 70 cm. high. No trees near. About sunset. Overcast with stratus. Observers : DAVID LONG and L. F. RICHARDSON.

III. 1917, June, 29d. 19h. 30m. L.A.T., Viel Dampierre, France. A field of corn 60 cm. high. Clouds (stratus) motionless. Observers : F. H. WEATHERALL, G. HUTCHINSON, L.F.R.

Eddy-stresses and of *Eddy-diffusivity* (continued).

$\widehat{yy}$ dynes per square centi-metre.	$\widehat{hh}$ dynes per square centi-metre.	$\widehat{xy}$ dynes per square centi-metre.	$\widehat{yh}$ dynes per square centi-metre.	$\widehat{hx}$ dynes per square centi-metre.	$\frac{\partial \bar{v}_x}{\partial h}$ sec.$^{-1}$.	Conductivity. $= \rho K$ grm. cm.$^{-1}$ sec.$^{-1}$.	Diffusivity. K cm.^{2} sec.$^{-1}$.	Turbulivity. $\xi = g^2\rho^2 K$ grm.^{2} cm.$^{-2}$ sec.$^{-5}$.
—	—	—	—	—	—	2·5	2,000	3,000
—	—	—	—	—	—	9·3	7,500	11,000
—	—	—	—	—	—	6	5,000	7,000
—	—	—	—	—	—	12	10,000	14,000
— —	-5 —							
—	—	—	—	—		120 —	10^5 —	$1 \cdot 4 \times 10^5$ —
—	—	—	—	—	—	$< 0 \cdot 9 \times 10^3$	$< 10^6$	$< 0 \cdot 8 \times 10^6$
-110								
—	—	—	—	—	—	0·12	100	140
—	—	—	—	—	—	12	10,000	14,000
—	-1							

IV. 1917, June, 25d. 20h. 5m. L.A.T., Joinville, France. Moor with herbage dense to 10 cm. and rising thinly to 50 cm.

V. 1917, October, 4d. $8\frac{1}{2}$h. G.M.T., Massiges, France. Flat moor with grass to 10 cm. and stems rising to 30 cm. Trees up-wind subtending an angle of 10 degrees. Overcast with strato nimbus, of which velocity/height = 0·025 sec.$^{-1}$. Temperature 0°·15 C. Observers: Olaf Stapledon and L. F. Richardson. Eddies partly due to observer.

VI. See photographs and description in this paper.

VII. 1917, July, 18d. $7\frac{1}{4}$h. L.A.T., Maffrecourt, France. Moor, with small trees, 5m. high and houses. Overcast.

VIII. 1917, August, 2d. 17h. L.A.T., South England. Overcast.

IX. 1917, July, 17d. 19h. 50m. L.A.T., East Champagne, France. Nine observations of a cloud at intervals of 1 minute by an Abney level. Angular elevation about 9 degrees.

X. 1917, August, 8d. 20h. L.A.T., Argonne, France. Large fire of petrol and wood. Smoke observed from distance of 10 km. with a sextant. If smoke were not hot, height of its upper edge above ground would have given $K = 4 \cdot 6 \times 10^5$ cm.^{2} sec.$^{-1}$, an over estimate. Irregularities in upper edge gave $K = 10^4$, probably an under estimate. Mean K of order of 10^5. Overcast.

XI. 1918, April, 12d. 14h. 5m. L.A.T., France. Two anti-aircraft shell puffs at a mean elevation of 21 degrees above the horizon and 4 degrees apart in a vertical plane, were brought into coincidence in the field of view of a sextant. In 120 seconds their separation of 4 degrees did not vary visibly, certainly not by 4 minutes of arc. Their apparent motion was horizontal at $\frac{1}{1800}$ radian per sec. Height *assumed* 3200 metres—a likely value.

XII. 1914, November, 16d. 3h. to 9h. G.M.T. Taken from the re-production of the Dines anemogram on p. 81 of the 'Observer's Handbook,' Meteorological Office, London, 1917 edition.

XIII. 1917, July, 26d. 21h. L.A.T., the English Channel.

XIV. 1917, July, 26d. 16h. L.A.T., the English Channel, off Havre.

XV. 1917, July, 26d. 15h. L.A.T., the English Channel, off Havre. Observations of cloud at intervals of 1 minute with pocket sextant. Angular elevation about 4 degrees.

The photograph (2) also shows that the smoke spreads more rapidly upwards than downwards, indicating that the stress $\widehat{hh}$ and the turbulivity ξ both increase with height.

Fig. 3 shows another case of low eddy-conductivity occurring at sunset. The smoke here is from burning hydrogen phosphides; it is warm and rises slightly. To the eye the smoke appeared as a narrow wavy ribbon moving with a mean velocity of 1·2 metres per second. The broader smooth band shown in the photograph is due to the exposure of 75 seconds, made long in order to get an average effect. The source of smoke is a bottle 3·4 metres above ground and just within the picture. The bamboos are 5 metres apart. The air density was 0·00126 c.g.s. The eddy-diffusivity works out to about 130 c.g.s. units, the eddy-conductivity to 0·16 c.g.s., the turbulivity to 200 c.g.s. The sky was cloudless. Obstructions to windward rose above the horizon to an angle of only $\frac{1}{50}$ radian. The photograph was taken in latitude 51° 37′ N., longitude 4m. 24s. west, at 1919, Sept., 29d. 17h. 58m. G.M.T.

Above is a table of observations. It is noticeable that when two of the direct stresses $\widehat{xx}$, $\widehat{yy}$, $\widehat{hh}$ have been measured at the same time and place, they have been found to be not very unequal. G. I. TAYLOR has published some observations which show the same thing. It is as though there were a kind of equipartition of energy between the three components of the eddying motion. A very marked increase in both direct stress and diffusivity takes place either with velocity or with height. A rapid increase of viscosity with height in the first 200 metres has also been deduced by W. SCHMIDT from wind observations made by HELLMANN over a piece of flat land. ('Sitz. Akad. Wiss.,' Wien, 1917, Heft 6, p. 17).

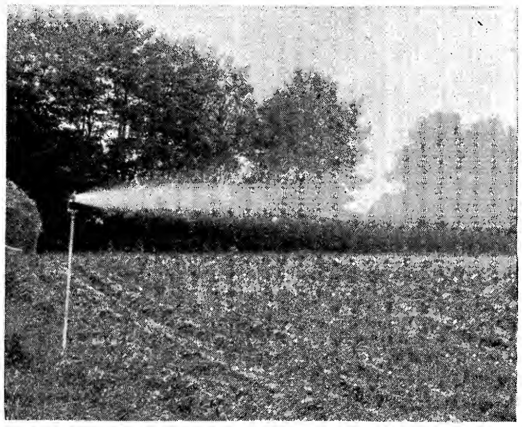


Fig. 1.

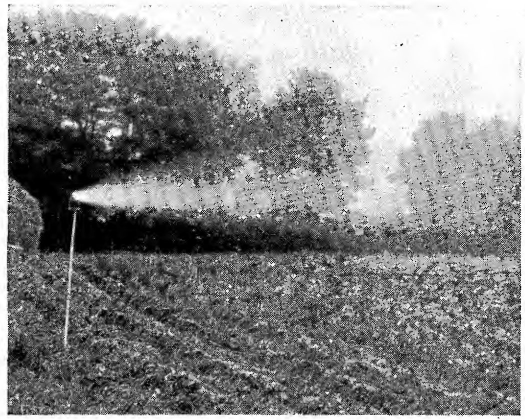
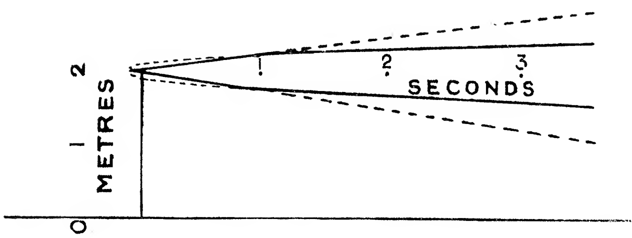
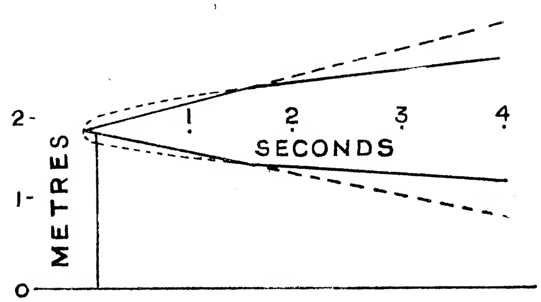


Fig 2.





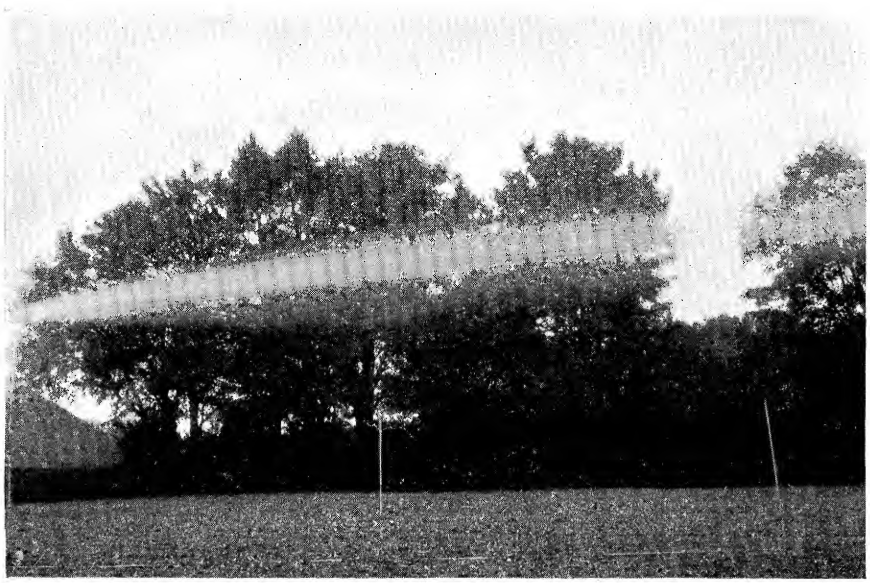


Fig. 3.

The six observations of steamers' smoke are put forward only as upper limits to the turbulence appropriate to the bare sea, for the steamer itself probably makes a considerable eddy.

It would be desirable to classify the observed eddy-conductivity as a function of four independent variables; namely, the height, the vertical gradient of entropy, the vertical gradient of velocity and the character of the surface. Vertical gradient of velocity is suggested as an independent because it measures the only rate-of-mean

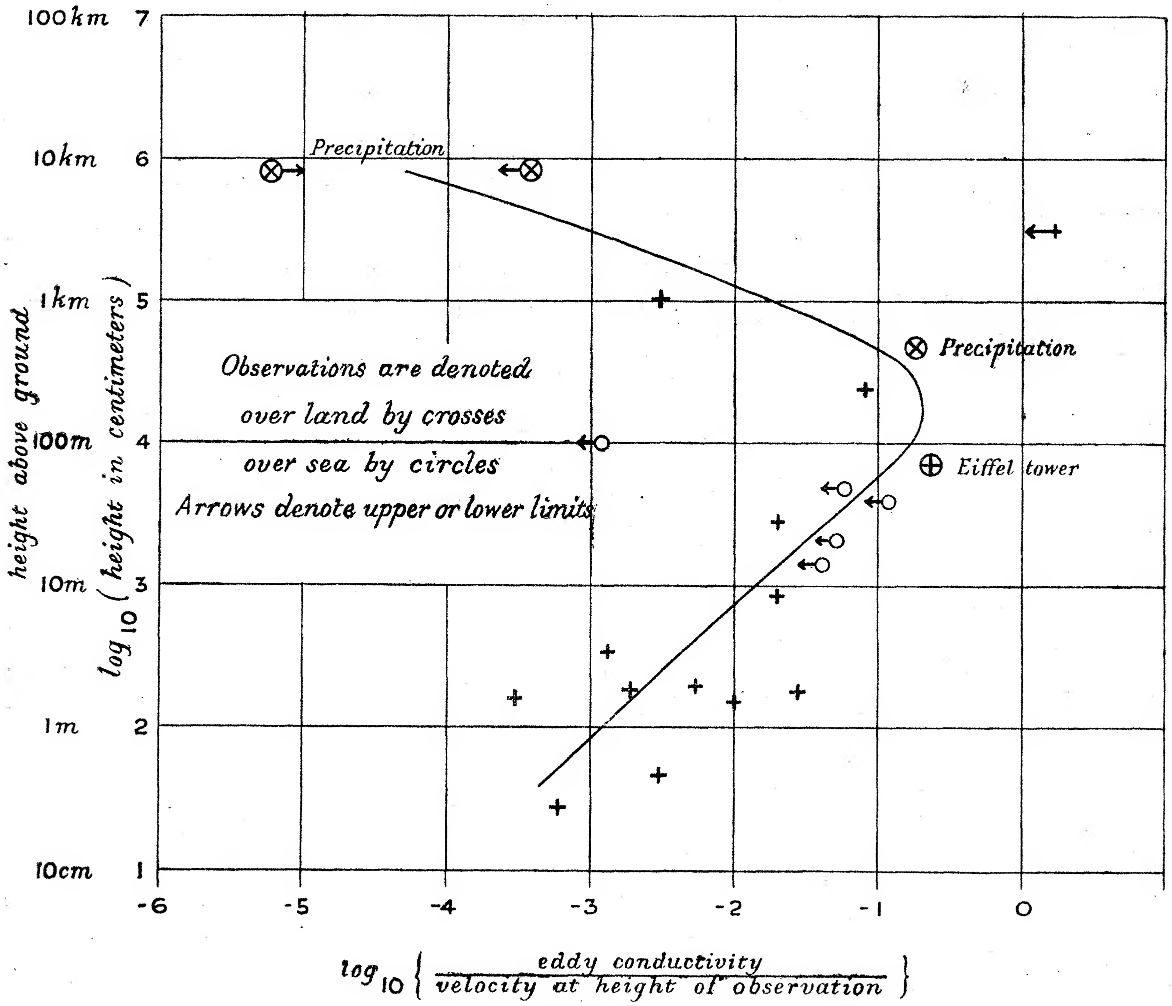


Fig. 4.

strain which attains a noticeable value in the free atmosphere, and because OSBORNE REYNOLDS* has shown that the energy of the eddy motion comes from the work done by the eddy-stresses upon the corresponding rates of mean strain. The observations here presented are much too scanty for such a classification, but to render the relation to height visible, the effect of velocity has been removed, in one sense, by dividing each value of the eddy-conductivity by the velocity at that level. The

* LAMB, 'Hydrodynamics,' IV. edition, § 369, equation (21).

justification for this procedure is that TAYLOR has given reasons* for supposing that the viscosity, and therefore also the conductivity, is proportional to the velocity. For comparison with the present observations, the diagram shows TAYLOR'S mean value of the diffusivity at the Eiffel Tower,† and also some general means‡ deduced from precipitation by the writer.

In order to compare them it has been necessary to assume some corresponding velocities; for which purpose I have taken 540 cm. sec.^{-1} at the mean height of the Eiffel Tower, 700 cm. sec.^{-1} as a world-mean at 500 metres and 1000 cm. sec.^{-1} for the same at 8500 metres. These are based on information given in HANN'S 'Meteorology.' The conversion formulæ between eddy-conductivity, diffusivity and ξ have been given in Section I. In order to compress into a diagram the large ranges of height and conductivity, logarithms have been plotted. A smooth curve is drawn through the clustered observations over land. It shows a maximum between the heights of 100 and 1000 metres, and a marked falling off above and below. Not only c/v but also c the conductivity has a maximum here. TAYLOR'S first observations related to heights near this maximum and so he naturally came to the conclusion that there was no marked variation with height.

IX. CUMULUS EDDIES IN CALM WEATHER.

The familiar sequence, which can be observed in many places, is here illustrated by the mean of some selected days in latitude 49° in France, on the bare grass moors to the west of the forest of Argonne, in the month of May. The sun rose at 4h. 20m. local apparent time, but could not be seen for mist. By 6h. the disk of the sun became visible. At $7\frac{1}{2}$h. the mist was rising in large pieces, leaving a brilliant blue sky. At 9h. the first cumuli appeared over the forest. About half-an-hour later they appeared over the grass land also. By noon the cumuli covered $\frac{4}{10}$ of the sky. By 16h. the cumuli had begun to spread out horizontally, and by 19h. they had vanished, leaving the sky clear again.

Now here we have a collection of eddies in which the rising parts, represented by the cumuli, visibly move to a level where they remain by mixing with their surroundings. So we should be able to calculate the diffusivity K by the direct application of the formula given by G. I. TAYLOR ('Phil. Trans.,' A, vol. 215, p. 3)

$$\mathrm{K} = \frac{1}{\mathrm{A}} \iint_{\mathrm{A}} v_{\mathrm{H}} (h-h')\, dx\, dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $h-h'$ is the height through which the air has moved before mixing, v_{H} is its vertical velocity, and A is a large horizontal area. Only, as TAYLOR'S formula assumes

* G. I. TAYLOR, 'Roy. Soc. Proc.,' A, vol. 92, pp. 196–199.

† G. I. TAYLOR, 'Roy. Soc. Proc.,' A, vol. 94 (1917), p. 141.

‡ L. F. RICHARDSON, 'Roy. Soc. Proc.,' A, vol. 96 (1919), p. 18.

that ρ and K are independent of height, it may be as well to remove these restrictions. It is then found that ξ, as defined by equation (1), of Part I., is given by

$$\xi = \frac{g}{\mathrm{A}} \iint_{\mathrm{A}} m_{\mathrm{H}} (p'-p)\, dx\, dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where p' is the pressure at the initial level, m_{H} the vertical momentum per volume.

Now let us insert numerical values. The surface air was seen to begin to move at or a little before $7\frac{1}{2}$h. The cumuli appeared to cease rising before 16h. and the height of their tops is known to average about 2 km. (*vide* HANN'S 'Meteorology,' IIIrd edn., p. 280). Now if we suppose, as seems reasonable, that the top of the cumulus is formed from the damp air which was initially close to the ground, then the displacement, measured by pressure, is about 2 decibars, so that $(p'-p) = 2 \times 10^5$ dyne cm.$^{-2}$. The vertical velocity is 2 km. in 8·5 hours, that is 6·5 cm. sec.$^{-1}$. So the momentum per volume $= m \ = 7 \times 10^{-3}$ grm. cm.$^{-2}$ sec.$^{-1}$ in the rising current. The rising current covered 0·4 of the sky, so that averaging over the area A, as is done in (2), is equivalent to taking 0·4 of $m_{\mathrm{H}}(p'-p)$ for the rising current. But the invisible descending currents contribute an equal amount to the integral. So

$$\begin{aligned} \xi &= g \times 0{\cdot}8 \times 2 \times 10^5 \times 7 \times 10^{-3} \\ &= 11 \times 10^5 \text{ grm.}^2 \text{ cm.}^{-2} \text{ sec.}^{-5}. \end{aligned}$$

This figure is about ten times greater than measures of ξ at a height of a few hundred meters, deduced by various authors. If the air which forms the top of the cumulus had really started from a height of 1 km. instead of from the ground, as we have supposed, then the numerical value of ξ would have to be divided by four.

Reasons have already been given (Part VII.) for supposing that ξ derived in this way from cumulus clouds is a measure of frictional effects, but not of the diffusion of entropy, because the linear term in (Part V., 3) does not vanish on taking the mean, owing to the fact that the eddies are produced by variations of entropy. To put it in another way: In G. I. TAYLOR'S deduction of formula (1) the vertical gradient of the diffusing quantity is treated as not correlated with the vertical velocity. When we are dealing with cumulus clouds that assumption is probably justified if the diffusing quantity is horizontal velocity, but not if it is potential temperature.

To find ξ in the sense of diffusivity for potential temperature we should have to employ formula 32 of Part VII., namely

$$\mathrm{K} = \frac{\overline{\left(\frac{\partial \Theta}{\partial h}\right)' v'_{\mathrm{H}}}}{\overline{\partial^2 \Theta / \partial h^2}}.$$

For insertion in this we require lapse rates in cumulus clouds and in the clear air between them. Such have recently been obtained by airmen.

X. Summary.

Part I. deals with notation. Measures of turbulence may advantageously be expressed in the form ξ in

$$\frac{\partial\chi}{\partial t} = \frac{\partial}{\partial p}\left(\xi\frac{\partial\chi}{\partial p}\right),$$

where p is the pressure (here used as a measure of height), t the time, and χ may be horizontal velocity in a fixed azimuth, or potential temperature, or water per mass of atmosphere. It is suggested that ξ might be called the "turbulivity." Its dimensions are grm.^{2} cm.$^{-2}$ sec.$^{-5}$. Better still is the conductivity $c = \xi\rho^{-1}g^{-2}$.

In Part II. the eddy-shearing stress on the ground is deduced from pilot balloon observations. Values on land in any self-consistent dynamical units are found to range from 0·0007 to 0·007 times the value of $\overline{m}^2/\rho$, where $\overline{m}$ is the mean momentum per volume up to a height of 2 km. and ρ is the density. Compare G. I. Taylor, 'Roy. Soc. Proc.,' A, vol. 92.

In Part III. evidence is given to show that the eddy-viscosity across the wind at Lindenberg increases with height, and, except near the ground, is much greater than the eddy-viscosity along the wind. Here ξ ranges from 10^4 to 5×10^5.

In Part IV. the spreading of a lamina of smoke is considered. Values of ξ ranging from 7 to 140,000 are found. ξ increases both with height and with velocity.

In Part V. the derivation of ξ from smoke observations is examined more thoroughly.

Part VI. deals with Osborne Reynolds' eddy-stresses. For one occasion an attempt was made to measure simultaneously all six components of stress by observing the motion of thistledown. The three direct stresses are easily measured. Not so the shearing stresses however, one was found to be 2·4 times its probable error.

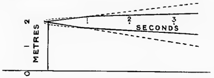
Part VII. summarizes the theory of scattering of particles.

Part VIII. contains numerical values derived from scattering.

In Part IX. the turbulivity ξ is estimated from the rising of cumuli in calm weather and found to be 10^6, applicable only in the sense of friction. Thus the whole range of ξ observed in the free atmosphere was from 7 to a million in contrast with 0·2 in perfectly still air in a laboratory. The eddy-stresses observed have ranged in absolute value from 0·004 to 110 dynes cm.$^{-2}$.

Fig. 1.

Fig 2.



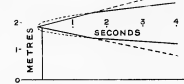


Fig. 3.